On a Method of Solving Equation of Radiative Transfer

著者	Tanaka Masayuki
雑誌名	Science reports of the Tohoku University. Ser. 5, Geophysics
巻	17
号	3
ページ	125-145
発行年	1966-03
URL	http://hdl.handle.net/10097/44673

On a Method of Solving an Equation of Radiative Transfer

MASAYUKI TANAKA

Geophysical Institute, Tohoku University, Sendai, Japan

(Received December 21, 1965)

Abstract: The equations for diffuse reflection and transmission in a finite plane-parallel atmosphere with an arbitrary stratification are solved by an iteration procedure. Expanding the scattering and transmission functions in the number of scatterings, we obtain the recurrence formulae for the functions representing the contributions due to respective orders of multiple scattering. Numerical results of an application of this method to a simple case show that the present method is suitable for the case in which either the atmosphere is thin or the albedo for single scattering is small.

1. Introduction

A method of solving the transfer equations for diffuse reflection and transmission in a finite plane-parallel atmosphere is discussed. In this problem, assuming a homogeneous stratified atmosphere, Chandrasekhar (1950) has developed a well-known method, in which the phase function is expanded in a finite number of Legendre polynomials and correspondingly the solution is expressed in terms of the X-and Y-functions. The X- and Y-functions in turn are defined as the solutions of a pair of simultaneous non-linear integral equations.

From the theoretical point of view this method is applicable to every cases. However, if the optical properties of the atmosphere are not simple and the phase function can not be expanded in a small number of Legendre polynomials, then we must consider many pairs of the X- and Y-functions, which are specified by different characteristic functions. In such a case the method of using the X- and Y-functions is not convenient.

In meteorological optics the atmosphere contains a large number of aerosol particles having different sizes and chemical compositions. The phase function changes its aspect according to the meteorological conditions, and in addition it can be determined only numerically from experiments by use of a polarnephelometer or calculation based on a scattering theory. In order to apply the Chandrasekhar method to these cases, it is necessary to find the best fitting coefficients of the expansion for a given phase function. Of course such a process itself is not so complicated, but in general it may be difficult to fit the given phase function in details by the expansion without using a large number of Legendre polynomials.

On the other hand Goldstein (1960) and Gross (1962) have studied, assuming an inhomogeneous and a homogeneous stratifications respectively, iterative solutions of the transfer equation for diffuse reflection in a semi-infinite plane-parallel atmosphere. In these studies, instead of the phase function, the scattering function is expanded in the number of scatterings, and the solution is obtained straightforwardly by recurrence formulae. This method adopted by Goldstein and Gross seems to be more suitable for the case of a finite atmosphere than that of semi-infinite atmosphere. Because less terms are needed in the expansion of the scattering function in the former case than in the latter one, owing to the fact that the contribution from the multi-scattered light increase with increase of the optical thickness.

In this paper, the same method as used by Goldstein (1960) and Gross (1962) is extensively applied to the transfer equations for diffuse reflection and transmission in a finite plane-parallel atmosphere with both homogeneous and inhomogeneous stratifications. The scattering and transmission functions are expanded in the number of scatterings and the recurrence formulae are derived for them. An alternative deduction of these formulae, which makes clear the physical ground of the method, is also discussed. Further, as a computational example, the application of the method to the simplest case of a homogeneous atmosphere with isotropic scattering is considered.

2. Integral equations for the scattering and transmission functions

The local optical properties of an atmosphere are represented by the phase function $p(\tau; \mu, \varphi; \mu', \varphi')$ which measures the fractional intensity per steradian scattered from the direction (μ', φ') into the direction (μ, φ). We can draw a line between an inhomogeneous and a homogeneous atmospheres according as the phase function p depends upon the optical thickness τ or not. We now give an outline of solutions for both atmospheres.

A homogeneous atmosphere. — In this case, a detailed discussion is not necessary. We shall denote the phase function by $p(\mu, \varphi; \mu', \varphi')$, the normalized phase function by $\gamma(\mu, \varphi; \mu', \varphi')$ and the albedo for single scattering by ω. The functions γ and ω are defined in the usual way:

$$\gamma(\mu, \varphi; \mu', \varphi') = \omega^{-1} p(\mu, \varphi; \mu', \varphi'), \tag{1}$$

and

$$\omega = \frac{1}{4\pi} \int_{-1}^{+1} \int_{0}^{2\pi} p(\mu, \varphi; \mu', \varphi')\, d\mu'\, d\varphi', \tag{2}$$

or equivalently

$$\omega = \beta^{(s)} / \beta^{(e)}, \tag{3}$$

where the volume scattering coefficient and the volume extinction (i.e., scattering plus absorption) coefficient of the medium are denoted by $\beta^{(s)}$ and $\beta^{(e)}$ respectively.

The global optical properties of the atmosphere, being the subjects of the problem of diffuse reflection and transmission, are expressed in terms of the scattering function $S(\tau_1; \mu, \varphi; \mu', \varphi')$ and the transmission function $T(\tau_1; \mu, \varphi; \mu', \varphi')$. These functions are defined as follows: If $I_{inc}(\mu', \varphi')$ represents the intensity of radiation incident in a plane-parallel atmosphere of optical thickness τ_1, in the direction $(-\mu', \varphi')$, the angular distribution of the intensity diffusely reflected from the surface $\tau=0$ and that diffusely transmitted below the surface $\tau=\tau_1$ are written in the forms

$$I(0; +\mu, \varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} S(\tau_1; \mu, \varphi; \mu', \varphi') I_{inc}(\mu', \varphi') \, d\mu' \, d\varphi', \tag{4}$$

and

$$I(\tau_1; -\mu, \varphi) = \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} T(\tau_1; \mu, \varphi; \mu', \varphi') I_{inc}(\mu', \varphi') \, d\mu' \, d\varphi'. \tag{5}$$

When a parallel beam of radiation of net flux πF per unit area normal to it is incident in the direction $(-\mu_0, \varphi_0)$, $I_{inc}(\mu', \varphi')$ is expressed in terms of Dirac's δ-functions in the form

$$I_{inc}(\mu', \varphi') = \pi F \, \delta(\mu' - \mu_0) \, \delta(\varphi' - \varphi_0). \tag{6}$$

Then, inserting (6) into (4) and (5), we get

$$I(0; +\mu, \varphi) = \frac{F}{4\mu} S(\tau_1; \mu, \varphi; \mu_0, \varphi_0), \tag{7}$$

and

$$I(\tau_1; -\mu, \varphi) = \frac{F}{4\mu} T(\tau_1; \mu, \varphi; \mu_0, \varphi_0). \tag{8}$$

Utilizing the transfer equation and the principle of invariance, Chandrasekhar (1950) has derived the following nonlinear intergral equations for the scattering and transmission functions:

$$\begin{aligned}
\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\tau_1; \Omega, \Omega_0) &= \omega \gamma(\Omega, -\Omega_0) \left[1 - \exp\left\{-\tau_1\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right\}\right] \\
&+ \frac{\omega}{4\pi} \int \gamma(\Omega, \Omega') S(\tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} + \frac{\omega}{4\pi} \int S(\tau_1; \Omega, \Omega') \gamma(-\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \\
&- \frac{\omega}{4\pi} e^{-\tau_1/\mu} \int \gamma(\Omega, -\Omega') T(\tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} \\
&- \frac{\omega}{4\pi} e^{-\tau_1/\mu_0} \int T(\tau_1; \Omega, \Omega') \gamma(\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \\
&+ \frac{\omega}{16\pi^2} \iint S(\tau_1; \Omega, \Omega') \gamma(-\Omega', \Omega'') S(\tau_1; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''} \\
&- \frac{\omega}{16\pi^2} \iint T(\tau_1; \Omega, \Omega') \gamma(\Omega', -\Omega'') T(\tau_1; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''},
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) T(\tau_1; \Omega, \Omega_0) &= \omega \gamma(-\Omega, -\Omega_0) \left\{e^{-\tau_1/\mu_0} - e^{-\tau_1/\mu}\right\} \\
&+ \frac{\omega}{4\pi} \int \gamma(-\Omega, -\Omega') T(\tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} - \frac{\omega}{4\pi} \int T(\tau_1; \Omega, \Omega') \gamma(-\Omega', -\Omega_0) \frac{d\Omega'}{\mu'}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\omega}{4\pi}\, e^{-\tau_1/\mu} \int \gamma(-\Omega, \Omega')\, S(\tau_1; \Omega', \Omega_0)\, \frac{d\,\Omega'}{\mu'} \\
&+ \frac{\omega}{4\pi}\, e^{-\tau_1/\mu_0} \int S(\tau_1; \Omega, \Omega')\, \gamma(\Omega', -\Omega_0)\, \frac{d\,\Omega'}{\mu'} \\
&+ \frac{\omega}{16\pi^2} \iint S(\tau_1; \Omega, \Omega')\, \gamma(\Omega', -\Omega'')\, T(\tau_1; \Omega'', \Omega_0)\, \frac{d\,\Omega'}{\mu'}\, \frac{d\,\Omega''}{\mu''} \\
&- \frac{\omega}{16\pi^2} \iint T(\tau_1; \Omega, \Omega')\, \gamma(-\Omega', \Omega'')\, S(\tau_1; \Omega'', \Omega_0)\, \frac{d\,\Omega'}{\mu'}\, \frac{d\,\Omega''}{\mu''}\,,
\end{aligned}
\tag{10}
$$

where (μ, φ) has been denoted by Ω and $(-\mu, \varphi)$ by $-\Omega$, and $\int_0^1 \int_0^{2\pi} \cdots\cdots d\,\mu'\, d\,\varphi'$ by $\int \cdots\cdots d\,\Omega'$.

An inhomogeneous atmosphere. — In this case, the phase function depends upon the optical thickness. We shall denote it by $p(\tau; \mu, \varphi; \mu', \varphi')$. The normalized phase function and the abledo for single scattering are defined by equations exactly similar to those for the case of the homogeneous atmosphere. We shall denote them by $\gamma(\tau; \mu, \varphi; \mu', \varphi')$ and $\omega(\tau)$ respectively.

It should be noted that τ-dependence of ω occurs, provided that the mixing ratio of absorbing and scattering particles contained in the atmosphere varies with altitude, while τ-dependence of γ occurs when the atmosphere contains more than two types of scattering particles, such as air molecules and aerosol particles, and when the mixing ratio of them varies with altitude.

In general, for a medium containing several types of scattering particles, each of them being denoted by an index i and characterized by a normalized phase function $\gamma_i\,(\mu, \varphi; \mu', \varphi')$ the functions p, γ and ω are expressed in the forms

$$
\begin{aligned}
p\,(\tau; \mu, \varphi; \mu', \varphi') &= \sum_i \omega_i\,(\tau)\, \gamma_i\,(\mu, \varphi; \mu', \varphi')\,, \\
\gamma\,(\tau; \mu, \varphi; \mu', \varphi') &= \sum_i \varpi_i\,(\tau)\, \gamma_i\,(\mu, \varphi; \mu', \varphi')\,, \\
\omega\,(\tau) &= \sum_i \omega_i\,(\tau)\,,
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
\omega_i\,(\tau) &= \beta_i{}^{(s)}/\beta^{(e)} = \beta_i{}^{(s)}/\sum_i \beta_i{}^{(e)}\,, \\
\varpi_i\,(\tau) &= \beta_i{}^{(s)}/\beta^{(s)} = \beta_i{}^{(s)}/\sum_i \beta_i{}^{(s)}\,.
\end{aligned}
\tag{12}
$$

The global optical properties of a semi-infinite inhomogeneous atmosphere which scatters radiation isotropically have been studied by Sobolev (1956) and others. Ueno (1960) has derived the complete set of the integral equations describing the global behavior of the radiation field in the problem of diffuse reflection and transmission in a finite plane-parallel atmosphere using the probabilistic method originated by Sobolev (1956) and also an extended invariance method. These methods give shape to an invaluable suggestion made by Preisendorfer (1958) that in the case of non-separable media, a pair of reflectance and transmittance operators for each of the two boundaries possesses polarity.

Although Ueno has treated in his study only the case of isotropic scattering, it is possible to apply these methods to the case of anisotropic scattering. Especially the application of the extended invariance method is appropriate in this case.

Now, along the line indicated by Ueno, we shall consider an inhomogeneous atmosphere of optical thickness $\tau_1 - \tau_0$ ($0 \leq \tau_0 \leq \tau_1$) illuminated in two ways by a parallel beam of radiation with net flux πF, i.e., illumination in the direction $(-\mu_0, \varphi_0)$ at the surface $\tau=\tau_0$ and in the direction (μ_0, φ_0) at the surface $\tau=\tau_1$. The scattering function $S(\tau_0, \tau_1; \mu, \mu_0)$ and the transmission function $T(\tau_0, \tau_1; \mu, \mu_0)$ in the case of uni-directional illumination of the surface $\tau=\tau_0$ are now replaced by $S(\tau_0, \tau_1; \mu, \varphi; \mu_0, \varphi_0)$ and $T(\tau_0, \tau_1; \mu, \varphi; \mu_0, \varphi_0)$ respectively, because of the azimuth dependence of the above functions. Similarly, the scattering function $S(\tau_1, \tau_0; \mu, \mu_0)$ and the transmission function $T(\tau_1, \tau_0; \mu, \mu_0)$ in the case of uni-directional illumination of the surface $\tau=\tau_1$ are replaced by $S(\tau_1, \tau_0; \mu, \varphi; \mu_0, \varphi_0)$ and $T(\tau_1, \tau_0; \mu, \varphi; \mu_0, \varphi_0)$, respectively.

The diffusely reflected and transmitted intensity for each case of the illuminations are expressed in terms of the corresponding pair of the scattering and transmission functions as follows:

$$I(\tau_0; +\mu, \varphi) = \frac{F}{4\mu} S(\tau_0, \tau_1; \mu, \varphi; \mu_0, \varphi_0), \tag{13}$$

$$I(\tau_1; -\mu, \varphi) = \frac{F}{4\mu} T(\tau_0, \tau_1; \mu, \varphi; \mu_0, \varphi_0), \tag{14}$$

for the case of uni-directional illumination of the upper boundary $\tau=\tau_0$, and

$$I^*(\tau_1; -\mu, \varphi) = \frac{F}{4\mu} S(\tau_1, \tau_0; \mu, \varphi; \mu_0, \varphi_0), \tag{15}$$

$$I^*(\tau_0; +\mu, \varphi) = \frac{F}{4\mu} T(\tau_1, \tau_0; \mu, \varphi; \mu_0, \varphi_0), \tag{16}$$

for the case of uni-directional illumination of the lower boundary $\tau=\tau_1$.

In a manner similar to that used by Ueno, starting with the principle of invariance and the transfer equation appropriate to each case of the illuminations and eliminating the intensity I or I^*, the complete set of the integral equations for the scattering and transmission functions are derived as follows:

$$S(\tau_0, \tau_1; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} \exp\left\{-(\tau-\tau_0)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\Big[p(\tau; \Omega, -\Omega_0) +$$

$$+ \frac{1}{4\pi}\int p(\tau; \Omega, \Omega') S(\tau, \tau_1; \Omega', \Omega_0)\frac{d\Omega'}{\mu'} +$$

$$+ \frac{1}{4\pi}\int S(\tau, \tau_1; \Omega, \Omega') p(\tau; -\Omega', -\Omega_0)\frac{d\Omega'}{\mu'} +$$

$$+ \frac{1}{16\pi^2} \iint S(\tau, \tau_1; \Omega, \Omega') p(\tau; -\Omega', \Omega'') S(\tau, \tau_1; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''} \Big] d\tau, \tag{17}$$

$$S(\tau_0, \tau_1; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} \exp\left\{-(\tau-\tau_0)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\} \Big[p(\tau; \Omega, -\Omega_0) +$$

$$+ e^{(\tau-\tau_0)/\mu_0} \frac{1}{4\pi} \int p(\tau; \Omega, -\Omega') T(\tau_0, \tau; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ e^{(\tau-\tau_0)/\mu} \frac{1}{4\pi} \int T(\tau, \tau_0; \Omega, \Omega') p(\tau; \Omega', -\Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ \exp\left\{(\tau-\tau_0)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\} \frac{1}{16\pi^2} \iint T(\tau, \tau_0; \Omega, \Omega') p(\tau; \Omega', -\Omega'')$$

$$\times T(\tau_0, \tau; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''} \Big] d\tau, \tag{18}$$

$$T(\tau_0, \tau_1; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} e^{-(\tau-\tau_0)/\mu_0} \Big[e^{-(\tau_1-\tau)/\mu} p(\tau; -\Omega, -\Omega_0) +$$

$$+ \frac{1}{4\pi} \int T(\tau, \tau_1; \Omega, \Omega') p(\tau; -\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ e^{-(\tau_1-\tau)/\mu} \frac{1}{4\pi} \int p(\tau; -\Omega, \Omega') S(\tau, \tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ \frac{1}{16\pi^2} \iint T(\tau, \tau_1; \Omega, \Omega') p(\tau; -\Omega', \Omega'') S(\tau, \tau_1; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''} \Big] d\tau, \tag{19}$$

$$T(\tau_0, \tau_1; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} e^{-(\tau_1-\tau)/\mu} \Big[e^{-(\tau-\tau_0)/\mu_0} p(\tau; -\Omega, -\Omega_0) +$$

$$+ \frac{1}{4\pi} \int p(\tau; -\Omega, -\Omega') T(\tau_0, \tau; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ e^{-(\tau-\tau_0)/\mu_0} \frac{1}{4\pi} \int S(\tau, \tau_0; \Omega, \Omega') p(\tau; \Omega', -\Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ \frac{1}{16\pi^2} \iint S(\tau, \tau_0; \Omega, \Omega') p(\tau; \Omega', -\Omega'') T(\tau_0, \tau; \Omega'', \Omega_0) \frac{d\Omega'}{\mu'} \frac{d\Omega''}{\mu''} \Big] d\tau, \tag{20}$$

$$S(\tau_1, \tau_0; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} \exp\left\{-(\tau_1-\tau)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\} \Big[p(\tau; -\Omega, \Omega_0) +$$

$$+ \frac{1}{4\pi} \int p(\tau; -\Omega, -\Omega') S(\tau, \tau_0; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+ \frac{1}{4\pi} \int S(\tau, \tau_0; \Omega, \Omega') p(\tau; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} +$$

$$+\frac{1}{16\pi^2}\iint S(\tau,\tau_0;\Omega,\Omega')\,p(\tau;\Omega',-\Omega'')\,S(\tau,\tau_0;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\Big]d\tau, \tag{21}$$

$$S(\tau_1,\tau_0;\Omega,\Omega_0)=\int_{\tau_0}^{\tau_1}\exp\left\{-(\tau_1-\tau)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\Big[p(\tau;-\Omega,\Omega_0)+$$

$$+e^{(\tau_1-\tau)/\mu_0}\frac{1}{4\pi}\int p(\tau;-\Omega,\Omega')\,T(\tau_1,\tau;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+e^{(\tau_1-\tau)/\mu}\frac{1}{4\pi}\int T(\tau,\tau_1;\Omega,\Omega')\,p(\tau;-\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+\exp\left\{(\tau_1-\tau)\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\frac{1}{16\pi^2}\iint T(\tau,\tau_1;\Omega,\Omega')\,p(\tau;-\Omega',\Omega'')$$

$$\times T(\tau_1,\tau;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\Big]d\tau, \tag{22}$$

$$T(\tau_1,\tau_0;\Omega,\Omega_0)=\int_{\tau_0}^{\tau_1}e^{-(\tau_1-\tau)/\mu_0}\Big[e^{-(\tau-\tau_0)/\mu}p(\tau;\Omega,\Omega_0)+$$

$$+\frac{1}{4\pi}\int T(\tau,\tau_0;\Omega,\Omega')\,p(\tau;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+e^{-(\tau-\tau_0)/\mu}\frac{1}{4\pi}\int p(\tau;\Omega,-\Omega')\,S(\tau,\tau_0;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+\frac{1}{16\pi^2}\iint T(\tau,\tau_0;\Omega,\Omega')\,p(\tau;\Omega',-\Omega'')\,S(\tau,\tau_0;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\Big]d\tau, \tag{23}$$

$$T(\tau_1,\tau_0;\Omega,\Omega_0)=\int_{\tau_0}^{\tau_1}e^{-(\tau-\tau_0)/\mu}\Big[e^{-(\tau_1-\tau)/\mu_0}p(\tau;\Omega,\Omega_0)+$$

$$+\frac{1}{4\pi}\int p(\tau;\Omega,\Omega')\,T(\tau_1,\tau;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+e^{-(\tau_1-\tau)/\mu_0}\frac{1}{4\pi}\int S(\tau,\tau_1;\Omega,\Omega')\,p(\tau;-\Omega',\Omega_0)\frac{d\Omega'}{\mu'}+$$

$$+\frac{1}{16\pi^2}\iint S(\tau,\tau_1;\Omega,-\Omega')\,p(\tau;-\Omega',\Omega'')\,T(\tau_1,\tau;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\Big]d\tau, \tag{24}$$

where (μ,φ), $(-\mu,\varphi)$ and $\int_0^1\int_0^{2\pi}\cdots\, d\mu'\,d\varphi'$ have been denoted again by Ω, $-\Omega$ and $\int\cdots\, d\Omega'$, respectively.

Although these eight equations describe the problem completely, only four among them are sufficient for ordinary problems, i.e., (17), (19) and (21), (23) which involve the pair of functions representing the forward and backward stochastic process of multiple scattering. In the case of isotropic scattering, the solutions of the corresponding set of the integral equations can be expressed in terms of the generalized X-and Y-functions (Ueno, 1960), the values of which are determined depending on the distribution of the albedo for single scattering, $\omega=\omega(\tau)$.

3. Expansion of the scattering and transmission functions

In this section, using a method similar to that used by Goldstein (1960) and Gross (1962), iterative solutions of the integral equations for the scattering and transmission functions are derived. Begining with the case of a homogeneous atmosphere, we expand the S- and T-functions in the power series of ω, which in this case is a given constant, in the forms

$$S(\tau_1; \Omega, \Omega_0) = \sum_{n=0}^{\infty} \omega^n S_{(n)}(\tau_1; \Omega, \Omega_0),$$

and (25)

$$T(\tau_1; \Omega, \Omega_0) = \sum_{n=0}^{\infty} \omega^n T_{(n)}(\tau_1; \Omega, \Omega_0).$$

Defining

$$\phi(\tau_1; \Omega, \Omega_0) = \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right) S(\tau_1; \Omega, \Omega_0),$$

$$\psi(\tau_1; \Omega, \Omega_0) = \left(\frac{1}{\mu} - \frac{1}{\mu_0}\right) T(\tau_1; \Omega, \Omega_0),$$

and expanding them, we get

$$\phi(\tau_1; \Omega, \Omega_0) = \sum_{n=0}^{\infty} \omega^n \phi_{(n)}(\tau_1; \Omega, \Omega_0), \tag{26}$$

$$\psi(\tau_1; \Omega, \Omega_0) = \sum_{n=0}^{\infty} \omega^n \psi_{(n)}(\tau_1; \Omega, \Omega_0),$$

where $\phi_{(n)}=\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)S_{(n)}$ and $\psi_{(n)}=\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)T_{(n)}$. Using (26) in place of (25) and inserting it into (9) and (10) and equating the corresponding coefficients on each side, the recursion relations for $\phi_{(n)}$ and $\psi_{(n)}$ are obtained as follows:

$$\phi_{(0)}(\tau_1; \Omega, \Omega_0) = 0, \tag{27-1}$$

$$\phi_{(1)}(\tau_1; \Omega, \Omega_0) = \gamma(\Omega, -\Omega_0)\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\right], \tag{27-2}$$

$$\phi_{(2)}(\tau_1; \Omega, \Omega_0) = \frac{1}{4\pi}\int \gamma(\Omega, -\Omega')\,\gamma(-\Omega', -\Omega_0)$$
$$\times\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\right\}\right]\frac{\mu}{\mu+\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}\int\gamma(\Omega,\Omega')\gamma(\Omega',-\Omega_0)\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu_0}+\frac{1}{\mu'}\right)\right\}\right]\frac{\mu_0}{\mu_0+\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}e^{-\tau_1/\mu_0}\int\gamma(\Omega,\Omega')\gamma(\Omega',-\Omega_0)\left[e^{-\tau_1/\mu}-e^{-\tau_1/\mu'}\right]\frac{\mu}{\mu-\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}e^{-\tau_1/\mu}\int\gamma(\Omega,-\Omega')\gamma(-\Omega',-\Omega_0)\left[e^{-\tau_1/\mu_0}-e^{-\tau_1/\mu'}\right]\frac{\mu_0}{\mu_0-\mu'}\,d\Omega', \tag{27-3}$$

$$\phi_{(n)}(\tau_1;\Omega,\Omega_0)=\frac{1}{4\pi}\int\phi_{(n-1)}(\tau_1;\Omega,\Omega')\gamma(-\Omega',-\Omega_0)\frac{\mu}{\mu+\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}\int\gamma(\Omega,\Omega')\phi_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{\mu_0}{\mu_0+\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}e^{-\tau_1/\mu_0}\int\psi_{(n-1)}(\tau_1;\Omega,\Omega')\gamma(\Omega',-\Omega_0)\frac{\mu}{\mu-\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}e^{-\tau_1/\mu}\int\gamma(\Omega,-\Omega')\psi_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{\mu_0}{\mu_0-\mu'}\,d\Omega'$$

$$+\frac{1}{16\pi^2}\sum_{m=1}^{n-2}\iint\phi_{(m)}(\tau_1;\Omega,\Omega')\gamma(-\Omega',\Omega'')\phi_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)$$

$$\times\frac{\mu}{\mu+\mu'}\,d\Omega'\,\frac{\mu_0}{\mu_0+\mu''}\,d\Omega''$$

$$+\frac{1}{16\pi^2}\sum_{m=1}^{n-2}\iint\psi_{(m)}(\tau_1;\Omega,\Omega')\gamma(-\Omega',\Omega'')\psi_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)$$

$$\times\frac{\mu}{\mu-\mu'}\,d\Omega'\,\frac{\mu_0}{\mu_0-\mu''}\,d\Omega'' \quad (\mathrm{n}\geq 3), \tag{27-4}$$

$$\psi_{(0)}(\tau_1;\Omega,\Omega_0)=0, \tag{28-1}$$

$$\psi_{(1)}(\tau_1;\Omega,\Omega_0)=\gamma(-\Omega,-\Omega_0)\left[e^{-\tau_1/\mu_0}-e^{-\tau_1/\mu}\right], \tag{28-2}$$

$$\psi_{(2)}(\tau_1;\Omega,\Omega_0)=\frac{1}{4\pi}e^{-\tau_1/\mu_0}\int\gamma(-\Omega,\Omega')\gamma(\Omega',-\Omega_0)$$

$$\times\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu'}\right)\right\}\right]\frac{\mu}{\mu+\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}e^{-\tau_1/\mu}\int\gamma(-\Omega,\Omega')\gamma(\Omega',-\Omega_0)\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu_0}+\frac{1}{\mu'}\right)\right\}\right]\frac{\mu_0}{\mu_0+\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}\int\gamma(-\Omega,-\Omega')\gamma(-\Omega',-\Omega_0)\left[e^{-\tau_1/\mu}-e^{-\tau_1/\mu'}\right]\frac{\mu}{\mu-\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}\int\gamma(-\Omega,-\Omega')\gamma(-\Omega',-\Omega_0)\left[e^{-\tau_1/\mu_0}-e^{-\tau_1/\mu'}\right]\frac{\mu_0}{\mu_0-\mu'}\,d\Omega', \tag{28-3}$$

$$\psi_{(n)}(\tau_1;\Omega,\Omega_0)=\frac{1}{4\pi}\int\psi_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',-\Omega_0)\frac{\mu}{\mu-\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}\int\gamma(-\Omega,-\Omega')\,\psi_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{\mu_0}{\mu_0-\mu'}\,d\Omega'$$

$$+\frac{1}{4\pi}e^{-\tau_1/\mu_0}\int\phi_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega_0)\frac{\mu}{\mu+\mu'}\,d\Omega'$$

$$-\frac{1}{4\pi}e^{-\tau_1/\mu}\int\gamma(-\Omega,\Omega')\,\phi_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{\mu_0}{\mu_0+\mu'}\,d\Omega'$$

$$+\frac{1}{16\pi^2}\sum_{m=1}^{n-2}\iint\phi_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega'')\,\psi_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)$$

$$\times\frac{\mu}{\mu+\mu'}\,d\Omega'\frac{\mu_0}{\mu_0-\mu''}\,d\Omega''$$

$$+\frac{1}{16\pi^2}\sum_{m=1}^{n-2}\iint\psi_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',\Omega'')\,\phi_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)$$

$$\times\frac{\mu}{\mu-\mu'}\,d\Omega'\frac{\mu_0}{\mu_0+\mu''}\,d\Omega''\qquad(n\geq 3)\,. \quad (28\text{-}4)$$

The functions $\phi_{(n)}$ and $\psi_{(n)}$ are determined, then, from (27) and (28) in terms of the prescribed phase function.

Next, the extension of the above method to the case of an inhomogeneous atmosphere will be discussed. In this case ω is no longer a constant but a function of τ. For the sake of convenience, we assume that $\omega(\tau)$ and all integrals involving this quantity are small although this assumption is not indispensable. We write

$$S(\tau_0,\tau_1;\Omega,\Omega_0)=\sum_{n=0}^{\infty}S_{(n)}(\tau_0,\tau_1;\Omega,\Omega_0)\,,$$
$$T(\tau_0,\tau_1;\Omega,\Omega_0)=\sum_{n=0}^{\infty}T_{(n)}(\tau_0,\tau_1;\Omega,\Omega_0)\,, \quad (29)$$

and

$$S(\tau_1,\tau_0;\Omega,\Omega_0)=\sum_{n=0}^{\infty}S_{(n)}(\tau_1,\tau_0;\Omega,\Omega_0)\,,$$
$$T(\tau_1,\tau_0;\Omega,\Omega_0)=\sum_{n=0}^{\infty}T_{(n)}(\tau_1,\tau_0;\Omega,\Omega_0)\,, \quad (30)$$

where the n-th term of the expansion is supposed to be the same order of magnitude as $[\omega(\tau)]^n$. Inserting (29) into (17) and (19) and equating terms of the same order of mangitude, we obtain

$$S_{(0)}(\tau_0,\tau_1;\Omega,\Omega_0)=0\,, \quad (31\text{-}1)$$

$$S_{(1)}(\tau_0,\tau_1;\Omega,\Omega_0)=\int_{\tau_0}^{\tau_1}p(\tau;\Omega,-\Omega_0)\,e^{(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\,d\tau$$

$$= \sum_i \gamma_i(\Omega, -\Omega_0) \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\, d\tau\,, \tag{31-2}$$

$$S_{(2)}(\tau_0, \tau_1; \Omega, \Omega_0) = \frac{1}{4\pi} \int_{\tau_0}^{\tau_1} e^{-(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\, d\tau$$

$$\times \left\{ \int p(\tau; \Omega, \Omega')\, S_{(1)}(\tau, \tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} \right.$$

$$\left. + \int S_{(1)}(\tau, \tau_1; \Omega, \Omega')\, p(\tau; -\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \right\}$$

$$= \frac{1}{4\pi} \sum_i \sum_j \int \gamma_i(\Omega, \Omega')\, \gamma_j(\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\, d\tau$$

$$\times \int_{\tau}^{\tau_1} \omega_j(\tau')\, e^{-(\tau'-\tau)\{(1/\mu')+(1/\mu_0)\}}\, d\tau'$$

$$+ \frac{1}{4\pi} \sum_i \sum_j \int \gamma_i(\Omega, -\Omega')\, \gamma_j(-\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \int_{\tau_0}^{\tau_1} \omega_j(\tau)\, e^{-(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\, d\tau$$

$$\times \int_{\tau}^{\tau_1} \omega_i(\tau')\, e^{-(\tau'-\tau)\{(1/\mu)+(1/\mu')\}}\, d\tau'\,, \tag{31-3}$$

$$S_{(n)}(\tau_0, \tau_1; \Omega, \Omega_0) = \frac{1}{4\pi} \sum_i \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)\{(1/\mu)+(1/\mu_0)\}}\, d\tau$$

$$\times \left\{ \int \gamma_i(\Omega, \Omega')\, S_{(n-1)}(\tau, \tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} \right.$$

$$+ \int S_{(n-1)}(\tau, \tau_1; \Omega, \Omega')\, \gamma_i(-\Omega', -\Omega_0) \frac{d\Omega'}{\mu'}$$

$$+ \frac{1}{4\pi} \sum_{m=1}^{n-2} \iint S_{(m)}(\tau, \tau_1; \Omega, \Omega')\, \gamma_i(-\Omega', \Omega'')\, S_{(n-m-1)}(\tau, \tau_1; \Omega'', \Omega_0)$$

$$\left. \times \frac{d\Omega''}{\mu''} \frac{d\Omega'}{\mu'} \right\} \qquad (n \geq 3)\,, \tag{31-4}$$

$$T_{(0)}(\tau_0, \tau_1; \Omega, \Omega_0) = 0\,, \tag{32-1}$$

$$T_{(1)}(\tau_0, \tau_1; \Omega, \Omega_0) = \int_{\tau_0}^{\tau_1} p(\tau; -\Omega, -\Omega_0)\, e^{-(\tau-\tau_0)/\mu_0}\, e^{-(\tau_1-\tau)/\mu}\, d\tau$$

$$= \sum_i \gamma_i(-\Omega, -\Omega_0) \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)/\mu_0}\, e^{-(\tau_1-\tau)/\mu}\, d\tau\,, \tag{32-2}$$

$$T_{(2)}(\tau_0, \tau_1; \Omega, \Omega_0)$$

$$= \frac{1}{4\pi} \int_{\tau_0}^{\tau_1} e^{-(\tau-\tau_0)/\mu_0}\, d\tau \left\{ \int T_{(1)}(\tau, \tau_1; \Omega, \Omega')\, p(\tau; -\Omega', -\Omega_0) \frac{d\Omega'}{\mu'} \right.$$

$$\left. + e^{-(\tau_1-\tau)/\mu} \int p(\tau; -\Omega, \Omega')\, S_{(1)}(\tau, \tau_1; \Omega', \Omega_0) \frac{d\Omega'}{\mu'} \right\}$$

$$= \frac{1}{4\pi} \sum_i \sum_j \int \gamma_i(-\Omega, -\Omega')\, \gamma_j(-\Omega', -\Omega_0) \frac{d\,\Omega'}{\mu'} \int_{\tau_0}^{\tau_1} \omega_j(\tau)\, e^{-(\tau-\tau_0)/\mu_0}\, d\,\tau$$

$$\times \int_{\tau}^{\tau_1} \omega_i(\tau')\, e^{-(\tau'-\tau)/\mu'}\, e^{(\tau_1-\tau')/\mu}\, d\,\tau'$$

$$+ \frac{1}{4\pi} \sum_i \sum_j \int \gamma_i(-\Omega, \Omega')\, \gamma_j(\Omega', -\Omega_0) \frac{d\,\Omega'}{\mu'} \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)/\mu_0}\, e^{-(\tau_1-\tau)/\mu}\, d\,\tau$$

$$\times \int_{\tau}^{\tau_1} \omega_j(\tau')\, e^{-(\tau'-\tau)\{(1/\mu')+(1/\mu)\}}\, d\,\tau', \qquad (32\text{-}3)$$

$$T_{(n)}(\tau_0, \tau_1; \Omega, \Omega_0) = \frac{1}{4\pi} \sum_i \int_{\tau_0}^{\tau_1} \omega_i(\tau)\, e^{-(\tau-\tau_0)/\mu_0}\, d\,\tau$$

$$\times \Big\{ \int T_{(n-1)}(\tau, \tau_1; \Omega, \Omega')\, \gamma_i(-\Omega', -\Omega_0) \frac{d\,\Omega'}{\mu'}$$

$$+ e^{-(\tau_1-\tau)/\mu} \int \gamma_i(-\Omega, \Omega')\, S_{(n-1)}(\tau, \tau_1; \Omega', \Omega_0) \frac{d\,\Omega'}{\mu'}$$

$$+ \frac{1}{4\pi} \sum_{m=1}^{n-2} \iint T_{(m)}(\tau, \tau_1; \Omega, \Omega')\, \gamma_i(-\Omega', \Omega'')\, S_{(n-m-1)}(\tau, \tau_1; \Omega'', \Omega_0)$$

$$\times \frac{d\,\Omega'}{\mu'} \frac{d\,\Omega''}{\mu''} \Big\} \qquad (n \geq 3), \qquad (32\text{-}4)$$

where ω_i, γ_i etc., defined by (11) and (12) have been used. It is possible to show that, in the special case of the homogeneous atmosphere, (31) and (32) are reduced to (27) and (28) respectively.

Inserting (30) into (21) and (23), similar expressions for the functions $S_{(n)}(\tau_1, \tau_0; \Omega, \Omega_0)$ and $T_{(n)}(\tau_1, \tau_0; \Omega, \Omega_0)$ are easily obtained. These relations recursively determine the functions $S_{(n)}$ and $T_{(n)}$; hence, if these series converge, solutions of the problem are obtained by (25), (29) and (30).

4. Alternative deduction; physical ground of the method

Although the results of the preceding section are satisfactory, the basis of the method by which they were derived is somewhat indistinct. For instance, we assumed there implicitly that ω, the albedo for single scattering, is smaller than unity. It is not easy to prove mathematically that the series introduced in the preceding section converge, and hence, represent the solutions in the entire domain, $0 \leq \omega \leq 1$, for an arbitrary phase function.

It appeared worth while, therefore, to make an effort to clarify the physical significance of the method and place the entire deduction on this ground. In this section, we shall show that it is possible to assign definite meanings to the functions $S_{(n)}$ and $T_{(n)}$, and ensure the convergence of the series for all values of ω on this physical ground. We shall consider here only the case of a homogeneous atmosphere. However, the method of deduction is quite similar and is easily extensible for general cases.

It is obvious that the intensity of diffusely reflected or trasmitted radiation is expressed as a sum of the intensities which are subject to various degrees of multiple scattering. We shall define the functions $M_{(n)}$ and $N_{(n)}$ by writting the contributions due to the n-times scattered radiation in the forms

$$I_{(n)}(0, +\Omega) = \frac{1}{4\pi\mu}\int M_{(n)}(\tau_1; \Omega, \Omega')\, I_{inc}(\Omega')\, d\,\Omega', \tag{33}$$

$$I_{(n)}(\tau_1, -\Omega) = \frac{1}{4\pi\mu}\int N_{(n)}(\tau_1; \Omega, \Omega')\, I_{inc}(\Omega')\, d\,\Omega'. \tag{34}$$

Suppose a parallel beam of radiation is incident. Substituting (6) in (33) and (34), we have

$$I_{(n)}(0, +\Omega) = \frac{F}{4\mu} M_{(n)}(\tau_1; \Omega, \Omega_0), \tag{35}$$

$$I_{(n)}(\tau_1, -\Omega) = \frac{F}{4\mu} N_{(n)}(\tau_1; \Omega, \Omega_0). \tag{36}$$

The equation of transfer for $I_{(n)}(\tau, \Omega)$ is written in the form

$$\frac{d\, I_{(n)}(\tau, \pm\Omega)}{d\,\tau} = \pm\frac{1}{\mu} I_{(n)}(\tau, \pm\Omega) \mp \frac{\omega}{4\pi\mu}\int \gamma(\pm\Omega, \Omega')\, I_{(n-1)}(\tau, \Omega')\, d\,\Omega'$$

$$\pm\frac{\omega}{4\pi\mu}\int \gamma(\pm\Omega, -\Omega')\, I_{(n-1)}(\tau, -\Omega')\, d\,\Omega' \qquad (0<\mu\leq 1), \tag{37}$$

where $I_{(0)}(\tau, +\Omega') = 0$ and $I_{(0)}(\tau, -\Omega') = \pi F e^{-\tau/\mu_0}\delta(\mu' - \mu_0)\,\delta(\varphi' - \varphi_0)$. Solutions of this equation are required to satisfy the following boundary conditions:

$$I_{(n)}(0, -\Omega) = 0 \quad \text{and} \quad I_{(n)}(\tau_1, +\Omega) = 0. \tag{38}$$

Using the functions defined above, we can formulate the principle of invariance for the n-times scattered radiation as follows:

$$\text{I.}\quad I_{(n)}(\tau, +\Omega) = \frac{F}{4\mu} e^{-\tau/\mu_0} M_{(n)}(\tau_1 - \tau; \Omega, \Omega_0)$$

$$+\frac{1}{4\pi\mu}\sum_{m=1}^{n-1}\int M_{(m)}(\tau_1 - \tau; \Omega, \Omega')\, I_{(n-m)}(\tau, -\Omega')\, d\,\Omega', \tag{39}$$

$$\text{II.}\quad I_{(n)}(\tau; -\Omega) = \frac{F}{4\mu} N_{(n)}(\tau; \Omega, \Omega_0)$$

$$+\frac{1}{4\pi\mu}\sum_{m=1}^{n-1}\int M_{(m)}(\tau; \Omega, \Omega')\, I_{(n-m)}(\tau, +\Omega')\, d\,\Omega', \tag{40}$$

$$\text{III.}\quad \frac{F}{4\mu} S_{(n)}(\tau_1; \Omega, \Omega_0) = \frac{F}{4\mu} M_{(n)}(\tau; \Omega, \Omega_0) + e^{-\tau/\mu} I_{(n)}(\tau, +\Omega)$$

$$+\frac{1}{4\pi\mu}\sum_{m=1}^{n-1}\int N_{(m)}(\tau; \Omega, \Omega')\, I_{(n-m)}(\tau, +\Omega')\, d\,\Omega', \tag{41}$$

$$\text{IV.}\quad \frac{F}{4\mu} N_{(n)}(\tau_1;\Omega,\Omega_0) = \frac{F}{4\mu} e^{-\tau/\mu_0} N_{(n)}(\tau_1-\tau;\Omega,\Omega_0)$$

$$+ e^{-(\tau_1-\tau)/\mu} I_{(n)}(\tau,-\Omega) + \frac{1}{4\pi\mu}\sum_{m=1}^{n-1}\int N_{(m)}(\tau_1-\tau;\Omega,\Omega') I_{(n-m)}(\tau,-\Omega')\,d\Omega'. \tag{42}$$

For the first order scattering, the last term of the above equations vanishes, so that for $n=1$ the principle of invariance is written in the simple forms as given by

$$I_{(1)}(\tau,+\Omega) = \frac{F}{4\mu} e^{-\tau/\mu_0} M_{(1)}(\tau_1-\tau;\Omega,\Omega_0), \tag{39'}$$

$$I_{(1)}(\tau,-\Omega) = \frac{F}{4\mu} N_{(1)}(\tau;\Omega,\Omega_0), \tag{40'}$$

$$\frac{F}{4\mu} S_{(1)}(\tau_1;\Omega,\Omega_0) = \frac{F}{4\mu} M_{(1)}(\tau;\Omega,\Omega_0) + e^{-\tau/\mu} I_{(1)}(\tau,+\Omega), \tag{41'}$$

$$\frac{F}{4\mu} T_{(1)}(\tau_1;\Omega,\Omega_0) = \frac{F}{4\mu} e^{-\tau/\mu_0} N_{(1)}(\tau_1-\tau;\Omega,\Omega_0) + e^{-(\tau_1-\tau)/\mu} I_{(1)}(\tau,-\Omega). \tag{42'}$$

Now, following the well known procedure initiated by Chandrasekhar (1950), which has been also applied in section 2 of this paper without any explanation, we can obtain a system of equations governing the function $M_{(n)}$ and $N_{(n)}$. We assume that all functions are continuously differentiable with respect to the optical thickness.

On differentiating (39), (40), (41) and (42) with respect to τ, approaching the variable τ either to the limit $\tau=0$ in (39) and (42) or to the limit $\tau=\tau_1$ in (40) and (41), and making use of the boundary conditions (38), we have after some calculation

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) M_{(1)}(\tau_1;\Omega,\Omega_0) = \omega\,\gamma(\Omega,-\Omega_0)\left[1-\exp\left\{-\tau_1\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right\}\right], \tag{43-1}$$

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) M_{(2)}(\tau_1;\Omega,\Omega_0)$$

$$= \frac{\omega}{4\pi}\int \gamma(\Omega,\Omega') M_{(1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'} + \frac{\omega}{4\pi}\int M_{(1)}(\tau_1;\Omega,\Omega')\gamma(-\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$- e^{-\tau_1/\mu}\frac{\omega}{4\pi}\int \gamma(\Omega,-\Omega') N_{(1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$- e^{-\tau_1/\mu_0}\frac{\omega}{4\pi}\int N_{(1)}(\tau_1;\Omega,\Omega')\gamma(\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}, \tag{43-2}$$

$$\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right) M_{(n)}(\tau_1;\Omega,\Omega_0)$$

$$= \frac{\omega}{4\pi}\int \gamma(\Omega,\Omega') M_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+\frac{\omega}{4\pi}\int M_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-e^{-\tau_1/\mu}\frac{\omega}{4\pi}\int\gamma(\Omega,-\Omega')\,N_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-e^{-\tau_1/\mu_0}\frac{\omega}{4\pi}\int N_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+\frac{\omega}{16\pi^2}\sum_{m=1}^{n-2}\iint M_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',\Omega'')\,M_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}$$

$$-\frac{\omega}{16\pi^2}\sum_{m=1}^{n-2}\iint N_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega'')\,N_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}\quad(n\geq 3), \tag{43-3}$$

$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)N_{(1)}(\tau_1;\Omega,\Omega_0)=\omega\,\gamma(-\Omega,-\Omega_0)\left[e^{-\tau_1/\mu_0}-e^{-\tau_1/\mu}\right], \tag{44-1}$$

$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)N_{(2)}(\tau_1;\Omega,\Omega_0)$$

$$=\frac{\omega}{4\pi}\int\gamma(-\Omega,-\Omega')\,N_{(1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-\frac{\omega}{4\pi}\int N_{(1)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-e^{-\tau_1/\mu}\frac{\omega}{4\pi}\int\gamma(-\Omega,\Omega')\,M_{(1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+e^{-\tau_1/\mu_0}\frac{\omega}{4\pi}\int M_{(1)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}, \tag{44-2}$$

$$\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)N_{(n)}(\tau_1;\Omega,\Omega_0)=\frac{\omega}{4\pi}\int\gamma(-\Omega,-\Omega')\,N_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-\frac{\omega}{4\pi}\int N_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$-e^{-\tau_1/\mu}\frac{\omega}{4\pi}\int\gamma(-\Omega,\Omega')\,M_{(n-1)}(\tau_1;\Omega',\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+e^{-\tau_0/\mu_0}\frac{\omega}{4\pi}\int M_{(n-1)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega_0)\frac{d\Omega'}{\mu'}$$

$$+\frac{\omega}{16\pi^2}\sum_{m=1}^{n-2}\iint M_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(\Omega',-\Omega'')\,N_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}$$

$$-\frac{\omega}{16\pi^2}\sum_{m=1}^{n-2}\iint N_{(m)}(\tau_1;\Omega,\Omega')\,\gamma(-\Omega',\Omega'')\,M_{(n-m-1)}(\tau_1;\Omega'',\Omega_0)\frac{d\Omega'}{\mu'}\frac{d\Omega''}{\mu''}$$

$$(n\geq 3)\,. \tag{44-3}$$

Defining $\phi_{(n)}=\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\omega^{-n}M_{(n)}$ and $\psi_{(n)}=\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)\omega^{-n}N_{(n)}$ and introducing them into (43–1)–(44–3), we find that these equations are identical with (27) and (28). Similarly, applying the above method to the case of an inhomogeneous atmosphere, we can also obtain the relations which are identical with (31) and (32).

From these results, it is clear now that the physical meaning of the method of the preceding section is to expand the radiation field, and then the related functions, according to the number of scatterings. On this ground and the uniqueness of the solution of the basic equation, we can conclude that the series (25), (29) and (30) converge and so represent the solutions uniqely in the entire domain, $0\leq\omega\leq 1$.

5. Example; rapidity of convergence of the solutions

We now consider, by way of a computational example, the application of our method to the case of homogeneous atmosphere with isotropic scattering — i.e., the case $\gamma(\Omega,\Omega_0)\equiv 1$ and ω is a constant with respect ot τ. This problem has been studied by many workers and exact solutions are available, which we shall use to compare with our results.

In this case the problem is independent of the variables φ and φ_0, and the intensities of diffusely reflected and transmitted radiations are written in the forms

$$I(0;+\mu)=\frac{F}{4\mu}S(\tau_1;\mu,\mu_0)=\frac{\mu_0 F}{4(\mu_0+\mu)}\sum\omega^n\phi_{(n)}(\tau_1;\mu,\mu_0),$$
$$I(\tau_1;-\mu)=\frac{F}{4\mu}T(\tau_1;\mu,\mu_0)=\frac{\mu_0 F}{4(\mu_0-\mu)}\sum\omega^n\psi_{(n)}(\tau_1;\mu,\mu_0). \tag{45}$$

The relations for the functions $\phi_{(n)}(\tau_1;\mu,\mu_0)$ and $\psi_{(n)}(\tau_1;\mu,\mu_0)$ are easily obtained from (27) and (28), and are not repeated here. To evaluate these functions, we adopt a direct numerical computation based on Gauss's quadrature formula dividing the integrals involved into 9 for the incident direction and 10 for the scattering direction.

Although the problem of diffuse reflection and transmission is mainly discussed with regard to the angular distribution of the intensities of emergent radiation we shall consider here the fluxes of emergent radiation in place of the intensities. Denoting the fluxes of diffusely reflected and transmitted radiations by $U(\tau_1,\mu_0)$, and $D(\tau_1,\mu_0)$ respectively and expanding them in the power series of ω, we have

$$U(\tau_1,\mu_0)=2\pi\int_0^1 I(0,+\mu)\,\mu\,d\mu=\sum\omega^n U_{(n)}(\tau_1,\mu_0),$$
$$D(\tau_1,\mu_0)=2\pi\int_0^1 I(\tau_1,-\mu)\,\mu\,d\mu=\sum\omega^n D_{(n)}(\tau_1,\mu_0), \tag{46}$$

where the n-th terms, $\omega^n U_{(n)}(\tau_1,\mu_0)$ and $\omega^n D_{(n)}(\tau_1,\mu_0)$, are the contributions to the total fluxes due to the n-times scattered radiation, and the functions $U_{(n)}(\tau_1,\mu_0)$ and $D_{(n)}(\tau_1,\mu_0)$ are written in the forms

$$U_{(n)}(\tau_1, \mu_0) = \pi F \mu_0 \frac{1}{2} \int_0^1 \phi_{(n)}(\tau_1; \mu, \mu_0) \frac{\mu}{\mu_0 + \mu} d\mu ,$$
$$D_{(n)}(\tau_1, \mu_0) = \pi F \mu_0 \frac{1}{2} \int_0^1 \psi_{(n)}(\tau_1; \mu, \mu_0) \frac{\mu}{\mu_0 - \mu} d\mu . \tag{47}$$

In evaluating U and D from (46) and (47), we use again Gauss's quadrature formula.

On the other hand, the exact solutions for U and D are expressed in terms of the X and Y-functions as follows:

$$U(\tau_1, \mu_0) = \pi F \mu_0 \frac{1}{2} \omega \int_0^1 [X(\mu_0) X(\mu) - Y(\mu_0) Y(\mu)] \frac{\mu}{\mu_0 + \mu} d\mu ,$$
$$D(\tau_1, \mu_0) = \pi F \mu_0 \frac{1}{2} \omega \int_0^1 [Y(\mu_0) X(\mu) - X(\mu_0) Y(\mu)] \frac{\mu}{\mu_0 - \mu} d\mu , \tag{48}$$

where

$$X(\mu) = 1 + \frac{\omega}{2} \mu \int_0^1 [X(\mu) X(\mu') - Y(\mu) Y(\mu')] \frac{d\mu'}{\mu + \mu'} ,$$
$$Y(\mu) = e^{-\tau_1/\mu} + \frac{\omega}{2} \mu \int_0^1 [Y(\mu) X(\mu') - X(\mu) Y(\mu')] \frac{d\mu'}{\mu + \mu'} . \tag{49}$$

We also evaluate U and D from (48) basing on the numerical values of the X-and

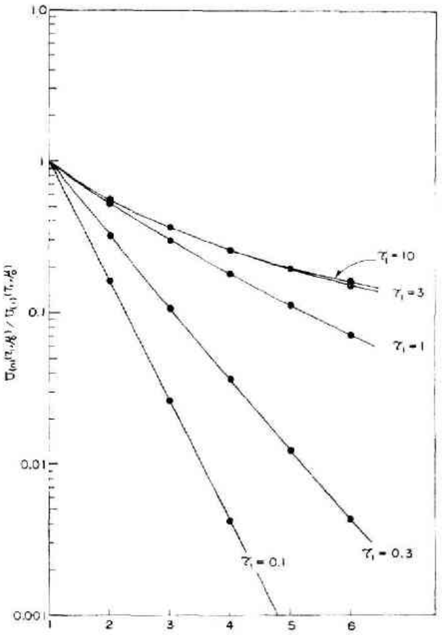

Fig. 1.

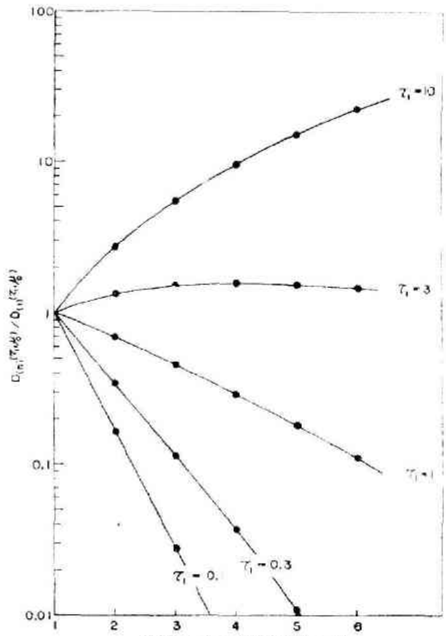

Fig. 2.

Fig. 1 Ratio $U_{(n)}(\tau_1, \mu_0)/U_{(1)}(\tau_1, \mu_0)$ as a function of n for $\mu_0 = 0.5$.
Fig. 2 Ratio $D_{(n)}(\tau_1, \mu_0)/D_{(1)}(\tau_1, \mu_0)$ as a function of n for $\mu_0 = 0.5$.

Y-functions tabulated by the Chandrasekhar et al. (1952) and Mayers (1962) and those obtained by our supplementary computations.

The relative importance of the radiation scattered once, twice, etc., are illustrated in Figs. 1–2. Fig 1 shows the ratio $U_{(n)}(\tau_1, \mu_0)$ to $U_{(1)}(\tau_1, \mu_0)$ as a function of n, the number of scatterings, taking τ_1 as a parameter. The direction of incidence, μ_0, is assumed to be 0.5. It will be seen in the figure that in the case of small τ_1 the ratio decreases rapidly with increase in n, but that in the case of large τ_1 the relative importance of the multi-scattered radiations increases. Fig. 2 shows, correspondingly, the ratio $D_{(n)}(\tau_1, \mu_0)$ to $D_{(1)}(\tau_1, \mu_0)$. The abscissa and the direction of incidence are the same as those of Fig. 1. When τ_1 is small the ratio decreases with increasing n as rapid as $U_{(n)}/U_{(1)}$. The ratio for higher values of n increases more rapidly than $U_{(n)}/U_{(1)}$ with increasing τ_1. The relative magnitude of the fluxes due to the n-times scattered radiation are given as the ratios, $U_{(n)}/U_{(1)}$ and $D_{(n)}/D_{(1)}$, multiplied by ω^{n-1}. For small values of ω only a few terms of the lower order are necessary. For large values of ω, however, heigher order terms are not negligible when the atmosphere becomes thick.

The diffusely reflected and transmitted fluxes for the cases of $\omega=1$, 0.6 and 0.2 are illustrated in Figs. 3–5. These figures show the approximate and exact solutions to the diffusely reflected flux (the left) and diffusely transmitted flux (the right) as a function of τ_1 or τ_1/μ_0, the effective thickness. The value of μ_0 is assumed to be 0.5. The curve denoted by n in the figure presents the n-th order approximate solution, i.e., the solution taking into account up to n-times scattered radiations, and the curve denoted by ∞ presents the exact solution. Although only one case of incidence, $\mu_0=0.5$, is illustrated, the figures for other values of μ_0 are essentially similar and roughly speaking they are independent of μ_0 if we use the effective thickness, τ_1/μ_0.

Comparing the approximate solutions of various orders with the exact solution, it is found, as expected from the physical grounds, that the present analysis is best applied to the atmosphere in which either τ_1 or ω is small. In a thick atmosphere with large ω,

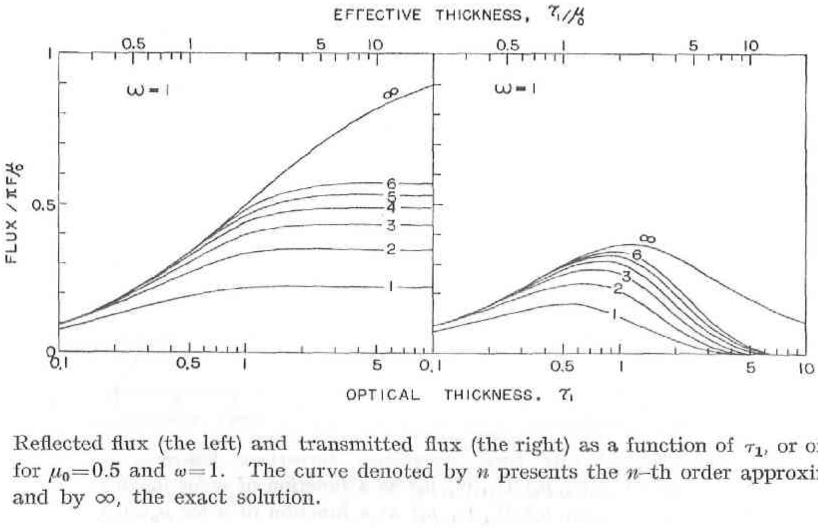

Fig. 3 Reflected flux (the left) and transmitted flux (the right) as a function of τ_1, or of τ_1/μ_0 for $\mu_0=0.5$ and $\omega=1$. The curve denoted by n presents the n-th order approximation, and by ∞, the exact solution.

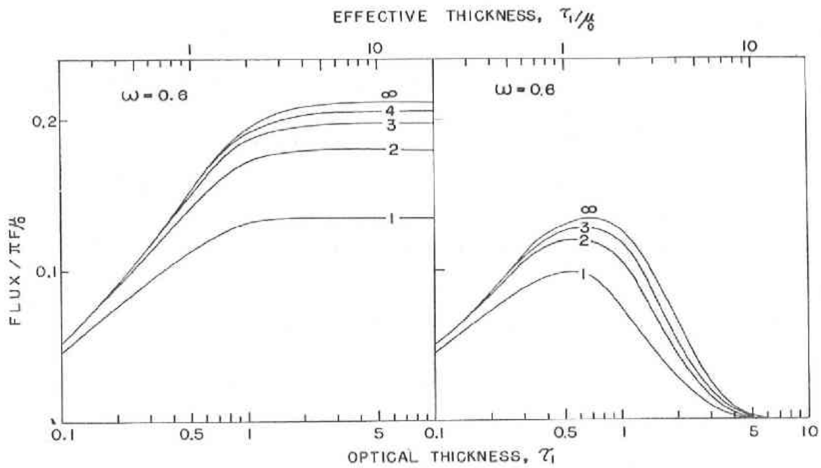

Fig. 4 Same as Fig. 3 but for $\omega=0.6$.

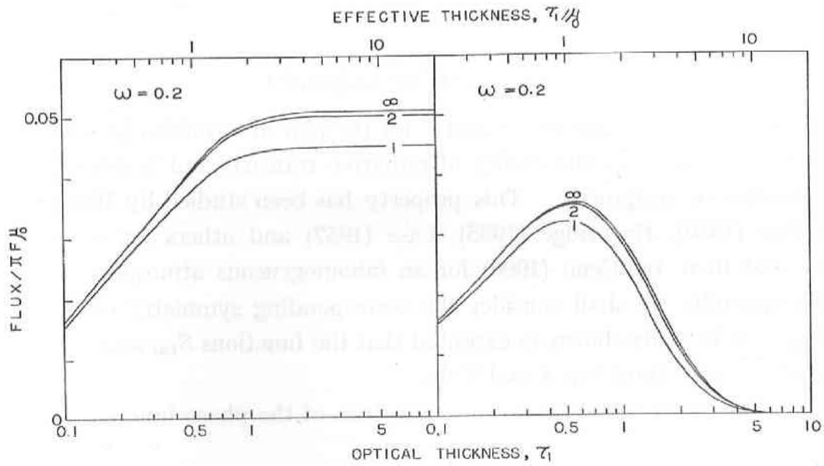

Fig. 5 Same as Fig. 3 but for $\omega=0.2$.

however, convergence of the series are not so rapid and in order to attain sufficient accuracy many terms must be taken. In this case it seems necessary to find other way of approach.

Acknowledgement: The author wishes to express his sincere thanks to Prof. G. Yamamoto for his kind guidance and encouragement throughout this work.

References

Busbridge, I.W., 1955: A mathematical verification of the principle of invariance as applied to completely non-coherent scattering and to interlocked multiplet lines. *Monthly Notices Roy. Astr. Soc.*, **115**, 661–670.

Case, K.M., 1957: Transfer problems and the reciprocity principle. *Rev. Mod. Phys.*, **29**, 651–663.

Chandrasekhar, S., 1950: "Radiative transfer." London Oxford Univ. Press. 393 pp.
Chandrasekhar, S., D. Elbert, and A. Franklin, 1952: The X and Y-function for isotropic scattering I. *Astrophys. J.*, **115**, 244–268.
Chandrasekhar, S., and D. Elbert, 1952: The X and Y-function for isotropic scattering II. *Astrophys. J.*, **115**, 269–278.
Goldstein, J.S., 1960: The infrared reflectivity of a planetary atmosphere. *Astrophys. J.*, **132**, 473–481.
Gross, K.I., 1962: Discussion of an iterative solution to an equation of radiative transfer. *J. Math. Phys.*, **41**, 53–61.
Mayers, D.F., 1962: Calculation of Chandrasekhar's X and Y functions for isotropic scattering. *Monthly Notices Roy. Astr. Soc.*, **123**, 471–484.
Minnaert, M., 1941: The reciprocity principle in lunar photometry. *Astrophys. J.*, **93**, 403–410.
Preisendorfer, R.W., 1958: Functional relations for the R and T operators on plane parallel media. *Proc. Nat. Acad. Sci. U.S.A.*, **44**, 323–327.
Sobolev, V.V., 1956: The transmission of radiation through an inhomogeneous medium. *Doklady Akad, Nauk, U.S.S.R.*, **111**, 1000–1003.
Ueno, S., 1960: The probabilistic method for problems of radiative transfer. X. Diffuse reflection and transmission in a finite inhomogeneous atmosphere. *Astrophys. J.*, **132**, 729–745.

Appendix

The principle of reciprocity

The symmetry of the functions S and T for the pair of variables (μ, φ) and (μ_0, φ_0) is an important property in the theory of radiative transfer, and is generally referred to as the principle of reciprocity. This property has been studied by Minnaert (1941), Chandrasekhar (1950), Busbridge (1955), Case (1957) and others for a homogeneous atmosphere and then, by Ueno (1960) for an inhomogeneous atmosphere.

In this appendix we shall consider the corresponding symmetry of the functions $S_{(n)}$ and $T_{(n)}$. It is easily shown as expected that the functions $S_{(n)}$ and $T_{(n)}$ follow the same symmetry as the functions S and T do.

We shall start from the well known symmetries of the phase functions as given by

$$\begin{aligned} &p(\tau; \Omega, \Omega_0) = p(\tau; \Omega_0, \Omega)\,, \\ &p(\tau; -\Omega, -\Omega_0) = p(\tau; \Omega, \Omega_0)\,, \\ &p(\tau; \Omega, -\Omega_0) = p(\tau; -\Omega_0, \Omega) = p(\tau; \Omega_0, -\Omega)\,. \end{aligned} \tag{A1}$$

Introducing (A1) into (43) and (44) and utilizing the method of mathematical deduction, the following relations are obtained for all values of $n \geq 1$:

$$\begin{aligned} &S_{(n)}(\tau_1; \Omega, \Omega_0) = S_{(n)}(\tau_1; \Omega_0, \Omega)\,, \\ &T_{(n)}(\tau_1; \Omega, \Omega_0) = T_{(n)}(\tau_1; \Omega_0, \Omega)\,. \end{aligned} \tag{A2}$$

Similarly, introducing (A1) into (31), (32) and corresponding relations for the functions $S_{(n)}(\tau_1, \tau_0; \Omega, \Omega_0)$ and $T_{(n)}(\tau_1, \tau_0; \Omega, \Omega_0)$ and making use of the method of mathematical deduction, we can also obtain the symmetry of the functions for an inhomogeneous atmosphere as follows:

$$S_{(n)}(\tau_0, \tau_1; \Omega, \Omega_0) = S_{(n)}(\tau_0, \tau_1; \Omega_0, \Omega),$$
$$S_{(n)}(\tau_1, \tau_0; \Omega, \Omega_0) = S_{(n)}(\tau_1, \tau_0; \Omega_0, \Omega), \tag{A3}$$

and

$$T_{(n)}(\tau_0, \tau_1; \Omega, \Omega_0) = T_{(n)}(\tau_1, \tau_0; \Omega_0, \Omega).$$

From definitions it is obvious that the functions S and T follow the same symmetry as the functions $S_{(n)}$ and $T_{(n)}$. It is easily shown that these relations are certainly compatible with the integral equations (9), (10) and (17)–(24).